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ON HERMITE-FEJÉR INTERPOLATION IN A JORDAN DOMAIN

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ABSTRACT. The Hermite-Fejér interpolation problem on a Jordan domain is studied. Under certain mild conditions on the smoothness of the boundary curve, we give both uniform and L^p, $0 < p < \infty$, estimates on the rate of convergence. Our estimates are sharp even for the unit disk setting.

1. INTRODUCTION

Let D be a Jordan domain in the complex plane $\mathbb{C}$ with boundary Γ and $z_k = z_{nk}$, $k = 1, \dots, n$, be sample points chosen on Γ. Also, let q be a nonnegative integer and $N = N_n := (q+1)n - 1$. In this paper we will consider the interpolation problem:

$$(1.1) \qquad \widetilde{H}_N(f; z_k) = f(z_k), \qquad \widetilde{H}_N^{(j)}(f; z_k) = a_k^{(j)},$$

$k = 1, \dots, n$ and $j = 1, \dots, q$, where f belongs to the class $A(\overline{D})$ of functions analytic in D and continuous on $\overline{D} = D \cup \Gamma$, and $\widetilde{H}_N(f; \cdot) \in \pi_N$, the space of all polynomials with degree at most N. Note that since f is not necessarily differentiable at z_k relative to $\overline{D}$ and the family of data values $\{a_k^{(j)}\}$ is arbitrarily given, the problem under consideration is different from the Hermite interpolation problem. In particular, by choosing $a_k^{(j)} = 0$ for all $k = 1, \dots, q$, the problem

$$(1.2) \qquad H_N(f; z_k) = f(z_k), \qquad H_N^{(j)}(f; z_k) = 0,$$

$k = 1, \dots, n$ and $j = 1, \dots, q$, where $f \in A(\overline{D})$ and $H_N(f; \cdot) \in \pi_N$, is usually called the $(0, 1, \dots, q)$ *Hermite-Fejér Interpolation Problem.*

It is well known that even for the unit disk $U = \{z : |z| < 1\}$, any q, and $z_k = e^{i2\pi k/n}$, there exists an $f \in A(\overline{U})$ such that $H_N(f; \cdot)$ does not converge uniformly on $\overline{U}$ to f (see [13]). In this paper, under certain smoothness conditions on the Jordan curve Γ, we will first give a necessary and sufficient

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condition on $f \in A(\overline{D})$ that guarantees uniform convergence of $\widetilde{H}_N(f;\cdot)$ to f on $\overline{D}$ for $a_n^{(j)} = o(n^j/\ln n)$, and derive the order of uniform convergence on $\overline{D}$ of the Hermite-Fejér interpolatory polynomials $H_N(f;\cdot)$ to f in terms of the modulus continuity of f. We will next show that for $a_n^{(j)} = o(n^j)$, $\widetilde{H}_N(f;\cdot)$ always converges in $L^p(\Gamma)$ to $f \in A(\overline{D})$, $0 < p < \infty$, and, in fact, a sharp order of convergence of $H_N(f;\cdot)$ in $L^p(\Gamma)$, $0 < p < \infty$, will be given.

Of course, for $q = 0$, problems (1.1) and (1.2) become the Lagrange interpolation problem:

$$L_N(f; z_k) = f(z_k), \tag{1.3}$$

$N = n - 1$, $k = 1, \ldots, n$, and $L_N \in \pi_N$. For an analytic Jordan curve Γ, Curtiss [3] has shown that $\|L_N(f;\cdot) - f\|_2 \to 0$ for all $f \in A(\overline{D})$ by using the Fejér nodes z_k on Γ. Here and throughout, $\|\cdot\|_p$ denotes the L^p-norm on Γ. Later, for a Jordan curve Γ of class $C^{2+\delta}$, for some $\delta > 0$, Al′per and Kalinogorskaja [2] improved the result in [3] by showing that

$$\|L_N(f;\cdot) - f\|_p \to 0$$

for any p, $0 < p < \infty$. Recently, this result was further improved by the second author and Zhong [10] to a Jordan curve Γ of class $C^{1+\delta}$ where the order of approximation $O(\omega(f; \frac{1}{N}))$ is also given. Here and throughout, $\omega(f;\delta)$ denotes the modulus of continuity of f on Γ using the *uniform norm*. We remark that the L^p, $0 < p < \infty$, modulus of continuity cannot be used even for the L^p estimate of $\|L_N(f;\cdot) - f\|_p$.

The only result in the literature for Hermite-Féjer interpolation on a Jordan curve different from the circle was obtained by Gaier [6], where an analytic curve Γ and $q = 1$ are considered and the convergence is only uniform on compact subsets of D. Various recent results concerning convergence on the unit disk of Hermite-Féjer interpolatory polynomials at the nth roots of unity can be found in Szabados and Varma [11], Varma [12], and the second author [8, 9].

2. Main results

Throughout this paper, $w = \Phi(z)$ denotes the exterior conformal map from $\mathbb{C}\backslash\overline{D}$ onto $|w| > 1$ such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Let $\Psi = \Phi^{-1}$ and write

$$z = \Psi(w) = dw + a_0 + a_1 w^{-1} + \cdots, \tag{2.1}$$

where $d = \Psi'(\infty) > 0$. It will be clear that by a standard transformation, we may assume, without loss of generality, that $d = 1$. Extend Ψ to a continuous function on $|w| \geq 1$ and set $z_k = z_{nk} = \Psi(w_{nk})$ where $w_{nk} = w_k = e^{i2\pi k/n}$. Recall that the z_{nk}'s are usually called the Fejér points on $\Gamma = \partial D$. We need some assumptions on the smoothness of Γ.

Definition. (i) Γ is said to be of class j_1 if $\Psi'(w)$ exists and is continuous on $|w| \geq 1$, and its (uniform) modulus of continuity $\sigma_1(t)$ on $|w| = 1$ satisfies the condition

$$\int_0^a \frac{\sigma_1(t)}{t} |\ln t|^2 \, dt < \infty, \qquad a > 0. \tag{2.2}$$

(ii) Γ is said to be of class j_2 if $\Psi''(w)$ exists and is continuous on $|w| \geq 1$, and its (uniform) modulus of continuity $\sigma_2(t)$ on $|w| = 1$ satisfies the condition

$$\int_0^a \frac{\sigma_2(t)}{t} |\ln t| \, dt < \infty, \qquad a > 0. \tag{2.3}$$

It is well known [1] that if Γ belongs to class j_1, then Ψ satisfies:

$$0 < C_1 \leq \left| \frac{\Psi(w) - \Psi(u)}{w - u} \right| \leq C_2 \tag{2.4}$$

for all $w \neq u$ and $|w|, |u| \geq 1$. We remark that in [1] it is shown that (2.4) already holds for those Γ with

$$\int_0^a \frac{\sigma_1(t)}{t} dt < \infty.$$

In addition, it is shown in the same paper that

$$0 < C_1 \leq |\Psi'(w)| \leq C_2 \tag{2.5}$$

for all w, $|w| \geq 1$.

Let

$$\omega_n(z) = \prod_{j=1}^{n} (z - z_j). \tag{2.6}$$

Then for each k, $(z - z_k)/\omega_n(z)$ is analytic at z_k, so that we can write

$$\left(\frac{z - z_k}{\omega_n(z)} \right)^{q+1} = \sum_{\nu=0}^{\infty} \alpha_{k\nu} (z - z_k)^{\nu}, \tag{2.7}$$

where $\alpha_{k\nu} = \alpha_{k\nu}(q, n)$, $q = 0, 1, \ldots$. In the following, we will give an asymptotic estimate of $\alpha_{k\nu} = \alpha_{k\nu}(q, n)$ as $n \to \infty$. We need the notation

$$\Omega_n(w) = \prod_{k=1}^{n} \frac{z - z_k}{w - w_k}, \qquad z = \Psi(w). \tag{2.8}$$

Theorem 1. *Let Γ belong to class j_2. Then for each ν and $q = 0, 1, \ldots$,*

$$\alpha_{k\nu} = \alpha_{k\nu}(q, n) = O\left(\frac{1}{n^{q+1-\nu}} \right) \tag{2.9}$$

and the estimate is uniform in k, $1 \leq k \leq n$, as $n \to \infty$.

Here and throughout, $\sum_{l \neq k}$ denotes the summation over all $l = 1, \ldots, n$ with $l \neq k$. To construct the interpolatory polynomials $\widetilde{H}_N(f; \cdot)$ and $H_N(f; \cdot)$

we introduce the *fundamental functions*:

$$A_{kj}(z) = \left(\frac{\omega_n(z)}{z - z_k}\right)^{q+1} \frac{(z - z_k)^j}{j!} \sum_{\nu=0}^{q-j} \alpha_{k\nu}(z - z_k)^\nu, \tag{2.10}$$

where $j = 0, \ldots, q$ and $l = 1, \ldots, n$. It is obvious that $A_{kj} \in \pi_N$ and we will verify that they satisfy

$$A_{kj}^{(\nu)}(z_l) = \delta_{kl}\delta_{\nu j}, \qquad k, l = 1, \ldots, n; \ \nu, j = 0, \ldots, q, \tag{2.11}$$

where, as usual, δ_{ij} denotes the Kronecker delta.

Theorem 2. *For any $f \in A(\overline{D})$, any nonnegative integer q, and arbitrary complex numbers $a_k^{(j)}$, $k = 1, \ldots, n$, $j = 1, \ldots, q$, there exists a unique $\widetilde{H}_N(f; \cdot) \in \pi_N$ satisfying the interpolation conditions* (1.1). *Furthermore, $\widetilde{H}_N(f; \cdot)$ is given by*

$$\widetilde{H}_N(f; \cdot) = \sum_{k=1}^{n} f(z_k)A_{k0}(\cdot) + \sum_{k=1}^{n}\sum_{j=1}^{q} a_k^{(j)} A_{kj}(\cdot). \tag{2.12}$$

In addition, under the assumption that Γ belongs to the class j_2, the fundamental functions A_{kj} satisfy the following estimates:

$$\max_{z\in\overline{D}} \sum_{k=1}^{n} |A_{kj}(z)| = O\left(\frac{\ln n}{n^j}\right), \qquad j = 0, \ldots, q, \tag{2.13}$$

and for $1 < p < \infty$,

$$\left\|\sum_{k=1}^{n} b_k A_{kj}(\cdot)\right\|_p = O\left(\frac{1}{n^j}\right) \max_{1\le k\le n} |b_k|, \qquad j = 0, \ldots, q, \tag{2.14}$$

for any sequence $\{b_k\}$, $k = 1, \ldots, n$.

Of course, if we choose $a_k^{(j)} = 0$, then the polynomials $\widetilde{H}_N(f; \cdot)$ become $H_N(f; \cdot)$ that satisfy the Hermite-Fejér interpolation condition (1.2). It is well known that even for the case $D = U$, the unit disk, there exists an $f \in A(\overline{U})$ such that $H_N(f; \cdot)$ does not converge uniformly to f on $\overline{U}$. We have the following result on the order of uniform approximation.

Theorem 3. *Let Γ belong to class j_2 and $f \in A(\overline{D})$. Then for any nonnegative integer q,*

$$\max_{z\in\overline{D}} |f(z) - H_N(f; z)| = O\left(\omega\left(f; \frac{1}{n}\right) \ln n\right). \tag{2.15}$$

We remark that this result is sharp as shown by the second author in [9] for $D = U$. For nonzero $a_k^{(j)}$, we have the following result.

Theorem 4. *Let* Γ *belong to class* j_2, $f \in A(\overline{D})$, *and* q *be any nonnegative integer. Suppose that*

$$\max_{1\le k\le n}\left|a_k^{(j)}\right| = o\left(\frac{n^j}{\ln n}\right), \qquad j = 1, \dots, q, \tag{2.16}$$

and

$$\lim_{\delta\to 0}\omega(f;\delta)\ln\delta = 0. \tag{2.17}$$

Then

$$\lim_{N\to\infty}\max_{z\in\overline{D}}\left|f(z) - \widetilde{H}_N(f;z)\right| = 0.$$

For L^p convergence, $0 < p < \infty$, we no longer need $\ln n$ in (2.15) as in the following

Theorem 5. *Let* Γ *belong to class* j_2, $f \in A(\overline{D})$, q *be any nonnegative integer, and* $0 < p < \infty$. *Then*

$$\|f - H_N(f;\cdot)\|_p = O\left(\omega\left(f;\frac{1}{n}\right)\right). \tag{2.18}$$

Again, this result is sharp even for $D = U$ as shown in [10]. For nonzero $a_k^{(j)}$, we have the following result.

Theorem 6. *Let* Γ *belong to class* j_2, $f \in A(\overline{D})$, q *be any nonnegative integer, and* $\{a_n^{(j)}\}$ *satisfy*

$$\max_{1\le k\le n}\left|a_k^{(j)}\right| = o(n^j), \qquad j = 1, \dots, q. \tag{2.19}$$

Then

$$\lim_{N\to\infty}\left\|f - \widetilde{H}_N(f;\cdot)\right\|_p = 0, \qquad 0 < p < \infty.$$

3. Proof of Theorem 1

To establish Theorem 1, we need three lemmas.

Lemma 1. *Let* Ψ'' *be continuous on* $|w| \ge 1$. *Then for each* $k = 1, \dots, n$,

$$\omega_n'(z_k) = n\frac{\Omega_n(w_k)}{\Psi'(w_k)w_k} \tag{3.1}$$

and

$$\sum_{l\ne k}\frac{1}{z_k - z_l} = \frac{1}{2}\frac{\omega_n''(z_k)}{\omega_n'(z_k)} \tag{3.2}$$

$$= \frac{1}{2\Psi'(w_k)}\left[\frac{n-1}{w_k} + \frac{2\Omega_n'(w_k)}{\Omega_n(w_k)} - \frac{\Psi''(w_k)}{\Psi'(w_k)}\right].$$

Proof. From (2.6) and (2.8), we have

$$\omega_n(z) = (w^n - 1)\Omega_n(w),$$

so that

$$\omega_n'(z) = [nw^{n-1}\Omega_n(w) + (w^n - 1)\Omega_n'(w)]\frac{1}{\Psi'(w)}, \tag{3.3}$$

from which (3.1) follows. To establish the two identities in (3.2), we first use logarithmic derivatives to obtain

$$\frac{\beta_k'(z)}{\beta_k(z)} = \sum_{l\neq k}\frac{1}{z-z_l}$$

with $\beta_k(z) := \omega_n(z)/(z-z_k)$. Since $\beta_k(z_k) = w_n'(z_k)$ and

$$\begin{aligned}\beta_k'(z_k) &= \lim_{z\to z_k}\frac{\omega_n'(z)(z-z_k)-\omega_n(z)}{(z-z_k)^2}\\ &= \lim_{z\to z_k}\frac{\omega_n'(z)(z-z_k)-\left[\omega_n'(z_k)(z-z_k)+\frac{\omega_n''(z_k)}{2}(z-z_k)^2+o(z-z_k)^2\right]}{(z-z_k)^2}\\ &= \lim_{z\to z_k}\frac{w_n''(z_k)(z-z_k)^2-\frac{w_n''(z_k)}{2}(z-z_k)^2+o(z-z_k)^2}{(z-z_k)^2} = \frac{1}{2}\omega_n''(z_k),\end{aligned}$$

we have established the first identity in (3.2). To derive the second identity in (3.2), we first observe that

$$\begin{aligned}\omega_n''(z_k) = n(n-1)w_k^{-2}\Omega_n(w_k) &+ 2nw_k^{-1}\Omega_n'(w_k)\frac{1}{[\Psi'(w_k)]^2}\\ &- nw_k^{-1}\Omega_n(w_k)\frac{\Psi''(w_k)}{[\Psi'(w_k)]^3}\end{aligned}$$

by using (3.3) and the fact that $w_k^n = 1$. By substituting this quantity and the quantity in (3.1) into $\omega_n''(z_k)/\omega_n'(z_k)$, we arrive at the second identity in (3.2). □

In the following, we give certain estimates on Ω_N and its relation with Ω_N'.

Lemma 2. *If* Γ *belongs to class* j_1, *then*

$$\max_{|w|\geq 1}|\Omega_n(w) - 1| = o\left(\frac{1}{\ln n}\right). \tag{3.4}$$

Furthermore, if Γ *belongs to class* j_2, *then*

$$\max_{|w|\geq 1}\left|\frac{\Omega_n'(w)}{\Omega_n(w)}\right| = o(1). \tag{3.5}$$

Proof. To prove (3.4), let

$$g(w, u) = \begin{cases}(\Psi(w)-\Psi(u))/(w-u) & \text{for } u\neq w,\\ \Psi'(w) & \text{for } u = w,\end{cases} \tag{3.6}$$

where $|u|, |w| \geq 1$. Hence, from the definition of $\Omega_N(w)$ and $g(w, u)$, we have

$$\ln \Omega_N(w) = \sum_{k=1}^{n} \ln g(w, w_k), \tag{3.7}$$

where the branch of the logarithm is taken so that $\ln 1 = 0$. On the other hand, it is clear that

$$\frac{\partial \ln g(w, u)}{\partial u} = \frac{-\Psi'(u)(w-u) + (\Psi(w) - \Psi(u))}{(w-u)(\Psi(w) - \Psi(u))} \tag{3.8}$$

and

$$\begin{aligned} \|\Psi(w) - \Psi(u) - \Psi'(u)(w-u)\| &= \left| \int_\gamma [\Psi'(\xi) - \Psi'(u)]\, d\xi \right| \\ &\leq C_1 \sigma_1(|w-u|) \int_\gamma |d\xi| \leq C_2 |w-u| \sigma_1(|w-u|), \end{aligned} \tag{3.9}$$

where γ is a contour joining u to w on $|\xi| \geq 1$ with length bounded by $\frac{\pi}{2}|u-w|$ and σ_1 denotes the modulus of continuity of Ψ'. By using (2.4), (3.8), and (3.9), we have

$$\left| \frac{\partial \ln g(w, u)}{\partial u} \right| \leq C \frac{\sigma_1(|w-u|)}{|w-u|},$$

for $|u|, |w| \geq 1$. Hence, from the hypothesis that Γ belongs to class j_1, as a function of u on $|u| = 1$, the function $\ln g(w, u)$ satisfies the Dini condition uniformly on $|w| = 1$. It follows that

$$\ln g(w, u) = \sum_{j=1}^{\infty} \frac{a_j(w)}{u^j} \tag{3.10}$$

uniformly on $|u|, |w| \geq 1$. From the property

$$\sum_{k=1}^{n} w_k^{-j} = \begin{cases} 0 & \text{if } n \nmid j, \\ n & \text{if } n \mid j \end{cases}$$

of the nth roots of unity, we have, from (3.7),

$$\ln \Omega_n(w) = n \sum_{l=1}^{\infty} a_{ln}(w) \tag{3.11}$$

uniformly on $|w| \geq 1$. To estimate $a_j(w)$, since Γ belongs to class j_1 we may use the Hardy-Littlewood inequality (cf. [4, p. 100])

$$|\Psi''(u)| \leq C \frac{\sigma_1(|u|-1)}{|u|-1}, \qquad |u| > 1. \tag{3.12}$$

Indeed, letting $1 < \rho \le \frac{3}{2}$, we have from (3.8), for $|w| = 1$,

$$\begin{aligned}
&\int_{|u|=\rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \right| |du| \\
&= \int_{|u|=\rho} \left| \frac{\Psi''(u)}{\Psi(u) - \Psi(w)} + \frac{(\Psi'(u))^2}{(\Psi(u) - \Psi(w))^2} - \frac{1}{(w-u)^2} \right| |du| \\
&\le \int_{|u|=\rho} \left| \frac{\Psi''(u)}{\Psi(n) - \Psi(w)} \right| |du| \\
&+ \int_{|u|=\rho} \left| \frac{\Psi'(u)}{\Psi(u) - \Psi(w)} - \frac{1}{w-u} \right| \left| \frac{\Psi'(u)}{\Psi(u) - \Psi(w)} + \frac{1}{w-u} \right| |du| \\
&:= I_1 + I_2,
\end{aligned}$$

where by applying (2.4) and (3.12), we have

$$I_1 \le C_1 \frac{\sigma_1(\rho - 1)}{\rho - 1} \int_{|u|=\rho} \frac{du}{|w-u|} \le C_2 \frac{\sigma_1(\rho-1)}{\rho-1} \ln \frac{1}{\rho - 1},$$

and by using (3.9) and (2.4), we also have

$$I_2 \le C_1 \int_{|u|=\rho} \frac{\sigma_1(|w-u|)}{|w-u|^2} |du| \le C_2 \frac{\sigma_1(\rho-1)}{\rho-1} \ln \frac{1}{\rho-1}.$$

That is,

$$\int_{|u|=\rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \right| |du| \le C \frac{\sigma_1(\rho-1)}{\rho-1} \ln \frac{1}{\rho-1}, \qquad \rho > 1. \tag{3.13}$$

By taking the second derivative of the power series (3.10) and applying the estimate in (3.13), we have, for $j = 2, 3, \dots$,

$$\begin{aligned}
|a_j(w)| &= \left| \frac{1}{2\pi i} \int_{|u|=\rho} \frac{1}{j(j+1)} \frac{\partial^2 \ln g(w, u)}{\partial u^2} u^{j+1} du \right| \\
&\le \frac{\rho^{j+1}}{2\pi j^2} \int_{|u|=\rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \right| |du| \le C \rho^{j+1} \frac{\sigma_1(\rho-1)}{j^2(\rho-1)} \ln \frac{1}{\rho-1}.
\end{aligned}$$

By taking $\rho = 1 + \frac{1}{j}$, it follows that

$$\max_{|w|=1} \left| a_j(w) \right| \le C \frac{\sigma_1(j^{-1})}{j} \ln j.$$

We now apply this estimate to (3.11), yielding

$$\begin{aligned}
\max_{|w|=1} |\ln \Omega_n(w)| &\le Cn \sum_{l=1}^{\infty} \frac{\sigma_1(1/ln)}{ln} \ln(ln) \\
&\le C \int_n^\infty \frac{\sigma_1(1/t)}{t} \ln t \, dt = C \int_0^{1/n} \frac{\sigma_1(s)}{s} |\ln s| \, ds \\
&\le \frac{C}{\ln n} \int_0^{1/n} \frac{\sigma_1(s)}{s} |\ln s|^2 ds = o\left(\frac{1}{\ln n}\right),
\end{aligned}$$

where (2.2) has been used. This estimate is equivalent to (3.4).

To prove (3.5) for Γ belonging to class j_2, we will apply the inequality of Hardy-Littlewood

$$|\Psi'''(u)| \le C\frac{\sigma_2(|u|-1)}{|u|-1}, \qquad |u|>1, \tag{3.14}$$

where σ_2 denotes the modulus of continuity of $\Psi''(u)$. Set

$$g_1(w, u) = \frac{\Psi'(u)}{\Psi(w)-\Psi(u)} - \frac{1}{w-u}.$$

Then by taking the logarithmic derivative of Ω_n and $g(w, w_k)$ in (2.8) and (3.6), respectively, we have

$$\frac{\Omega_n'(w)}{\Omega_n(w)} = \sum_{k=1}^{n} \frac{g'(w, w_k)}{g(w, w_k)} = \sum_{k=1}^{n} g_1(w, w_k), \tag{3.15}$$

where $g_1(w, u) = \Psi'(u)/(\Psi(w)-\Psi(u)) - 1/(w-u)$, hence, in view of Γ belonging to class j_2, we have

$$g_1(w, u) = \sum_{j=1}^{\infty} \frac{b_j(w)}{u^j} \tag{3.16}$$

uniformly on $|u|, |w| \ge 1$. By applying (3.14) we may obtain an estimate similar to that of (3.13), namely,

$$\int_{|u|=\rho} \left|\frac{\partial^2 g_1(w, u)}{\partial u^2}\right| |du| \le C\frac{\sigma_2(\rho-1)}{\rho-1} \ln\frac{1}{\rho-1},$$

where $\rho > 1$. Hence, as before, we have

$$\max_{|w|=1} \left|b_j(w)\right| \le C\frac{\sigma_2(j^{-1})}{j} \ln j$$

and

$$\frac{\Omega_n'(w)}{\Omega_n(w)} = n\sum_{l=1}^{\infty} b_{ln}(w), \qquad |w|=1,$$

so that

$$\max_{|w|\ge 1} \left|\frac{\Omega_n'(w)}{\Omega_n(w)}\right| \le Cn\sum_{l=1}^{\infty} \frac{\sigma_2(1/ln)}{ln} \ln(ln) \le \int_0^{1/n} \frac{\sigma_2(t)}{t} |\ln t|\, dt$$

which is $o(1)$ by (2.3). This completes the proof of the lemma.

Remark. In [10], where Lagrange interpolation (or $q=0$) was considered, the Jordan curve Γ was assumed to belong to $C^{1+\delta}$, $\delta > 0$. However, from our estimate (3.4) and the procedure in [10], it can be shown that the result there also holds for Γ belonging to class j_1.

As a consequence of estimates (3.5) in Lemma 2, the identity (3.2) in Lemma 1 yields the following result.

Corollary 1. *Let* Γ *belong to class* j_2. *Then*

$$\sum_{l\neq k}\frac{1}{z_k-z_l}=\frac{(n-1)}{2w_k\Psi'(w_k)}-\frac{\Psi''(w_k)}{2\left[\Psi'(w_k)\right]^2}+o(1) \tag{3.17}$$

uniformly in k, $1\leq k\leq n$.

In the proof of Theorem 1, the following estimates will also be used.

Lemma 3. *Let* Γ *belong to* j_2. *Then for* $k=1,2,\ldots,n$ *and* $r=0,1,\ldots$

$$\sum_{l\neq k}\frac{1}{(z_k-z_l)^{r+1}}=O(n^{r+1}) \tag{3.18}$$

and

$$\frac{d^r}{dz^r}\left(\frac{z-z_k}{\omega_n(z)}\right)\Bigg|_{z=z_k}=O(n^{r-1}) \tag{3.19}$$

uniformly in k, $1\leq k\leq n$.

Proof. The estimate (3.18) for $r=0$ can easily be deduced by (3.17). For $r\geq 1$, by using (2.4) we have

$$\begin{aligned}\left|\sum_{l\neq k}\frac{1}{(z_k-z_l)^{r+1}}\right| &\leq \sum_{l\neq k}\left|\frac{1}{z_k-z_l}\right|^{r+1}\leq C_2\sum_{l\neq k}\left|\frac{1}{w_k-w_l}\right|^{r+1}\\ &\leq 2C_2\sum_{l=1}^{[(n-1)/2]}\frac{1}{(2\sin l\pi/n)^{r+1}}\\ &\leq 2C_2\sum_{l=1}^{[(n-1)/2]}\frac{1}{(4l/n)^{r+1}}=O(n^{r+1}).\end{aligned}$$

We are going to verify (3.19) by induction. For $r=0$, by using (3.1), (3.4), and (2.5), we have (3.19). For $r\geq 1$, by the induction hypothesis and using (3.18), we obtain

$$\begin{aligned}\left(\frac{z-z_k}{\omega_n(z)}\right)^{(s+1)}\Bigg|_{z=z_k} &= -\left(\frac{z-z_k}{\omega_n(z)}\sum_{\substack{l=1\\ l\neq k}}^{n}\frac{1}{z-z_l}\right)^{(s)}\Bigg|_{z=z_k}\\ &= -\sum_{\nu=0}^{s}\binom{s}{\nu}\left(\frac{z-z_k}{\omega_n(z)}\right)^{(\nu)}\left(\sum_{\substack{l=1\\ l\neq k}}^{n}\frac{1}{z-z_l}\right)^{(s-\nu)}\Bigg|_{z=z_k}\\ &= -\sum_{\nu=0}^{s}\binom{s}{\nu}\left(\frac{z-z_k}{\omega_n(z)}\right)^{(\nu)}\sum_{\substack{l=1\\ l\neq k}}^{n}\frac{(-1)^{s-\nu}(s-\nu)!}{(z-z_l)^{s-\nu+1}}\Bigg|_{z=z_k}\\ &= \sum_{\nu=0}^{n}\binom{s}{\nu}O(n^{n-1})\cdot O\left(n^{s-\nu+1}\right)=O(n^{s}).\end{aligned}$$

This completes the proof of the lemma. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. For $q = 0$, from (2.7), it is clear that

$$\sigma_{k\nu}(0) = \frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left(\frac{z - z_k}{\omega_n(z)}\right)\Bigg|_{z=z_k}$$

which yields (2.9) by using (3.19) in Lemma 3. We will now use induction in q. Indeed, by the induction hypothesis and using (3.19) in Lemma 3 it follows that

$$\begin{aligned}\sigma_{k\nu}(q+1) &= \frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left(\frac{z - z_k}{\omega_n(z)}\right)^{q+2}\Bigg|_{z=z_k} \\ &= \frac{1}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}\frac{d^j}{dz^j}\left(\frac{z - z_k}{\omega_n(z)}\right)^{q+1}\frac{d^{\nu-j}}{dz^{\nu-j}}\left(\frac{z - z_k}{\omega_n(z)}\right)\Bigg|_{z=z_k} \\ &= \frac{1}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j} j! O\left(\frac{1}{n^{q-j+1}}\right) O\left(n^{\nu-j-1}\right) = O\left(\frac{1}{n^{q+2-\nu}}\right).\end{aligned}$$

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

We first establish the existence and uniqueness of $\widetilde{H}_N(f;\cdot)$ for any given $f \in A(\overline{D})$. Since $N = (q+1)n - 1$, it follows from the definition (2.11) that $A_{k_j} \in \pi_N$. Hence, $\widetilde{H}_N(f;\cdot) \in \pi_N$ also. Next, we will establish (2.11). For $l \neq k$, it is clear from the first factor that $A_{kj}^{(\nu)}(z_l) = 0$ for all ν, $j = 0, \dots, q$. We now consider the case $l = k$. From (2.10) and (2.7), it follows that

$$\begin{aligned}A_{kj}(z) &= \left(\frac{\omega_n(z)}{z - z_k}\right)^{q+1}\frac{(z-z_k)^j}{j!}\left[\left(\frac{z - z_k}{\omega_n(t)}\right)^{q+1} - \sum_{\mu=q-j+1}^{\infty}\alpha_{k\mu}(z-z_k)^\mu\right] \\ &= \frac{(z-z_k)^j}{j!} - \left(\frac{\omega_n(z)}{z - z_k}\right)^{q+1}\frac{1}{j!}\sum_{\mu=q-j+1}^{\infty}\alpha_{k\mu}(z-z_k)^{\mu+j}.\end{aligned}$$

Hence, for $\nu < j$, we have $A_{kj}^{(\nu)}(z_k) = 0$. For $\nu = j$, then $A_{kj}^{(j)}(z_k) = 1$. Finally, for $\nu > j$, we also have $A_{kj}^{(\nu)}(z_k) = 0$. This establishes the interpolatory property of A_{kj} in (2.11). Thus, by defining $\widetilde{H}_N(f;\cdot)$ as in (2.12), $\widetilde{H}_N(f;\cdot)$ solves the interpolation problem (1.1). The uniqueness of $\widetilde{H}_N(f;\cdot)$ is trivial. □

In order to establish the estimates (2.13) and (2.14), we need the following lemma.

Lemma 4. *Let* Γ *belong to class* j_1. *Then*

$$\max_{z\in\overline{D}}\sum_{k=1}^{n}\left|\frac{\omega_n(z)}{(z-z_k)\omega_n'(z_k)}\right| = O(\ln n), \tag{4.1}$$

$$\max_{z\in\overline{D}}\sum_{k=1}^{n}\left|\frac{\omega_n(z)}{(z-z_k)\omega_n'(z_k)}\right|^{1+\delta} = O(1), \qquad \delta>0, \tag{4.2}$$

and for $1<p<\infty$,

$$\left\|\sum_{k=1}^{n} b_k \frac{\omega_n(z)}{(z-z_k)\omega_n'(z_k)}\right\|_p = O\left(\max_{1\le k\le n}|b_k|\right). \tag{4.3}$$

Proof. From (2.8) and (3.1), we have

$$\frac{\omega_n(z)}{(z-z_k)\omega_n'(z_k)} = \frac{w_k\Psi'(w_k)(w^n-1)}{n(\Psi(w)-\Psi(w_k))}\,\frac{\Omega_n(w)}{\Omega_n(w_k)}, \tag{4.4}$$

where $z=\Psi(w)$ and $z_k=\Psi(w_k)$. For the unit disk, it is well known (cf. Gaier [6, pp. 80–81]) that

$$\max_{|w|\le 1}\sum_{k=1}^{n}\left|\frac{w_k}{n}\,\frac{w^n-1}{w-w_k}\right| = O(\ln n).$$

In addition, for $\delta>0$ it is also well known (cf. [9]) that

$$\max_{|w|\le 1}\sum_{k=1}^{n}\left|\frac{w_k}{n}\,\frac{w^n-1}{w-w_k}\right|^{1+\delta} = O(1).$$

Hence, by applying (2.4), (2.5), and (3.4) in Lemma 2 to (4.4), we have both (4.1) and (4.2). Next, by (4.4), it follows that

$$\begin{aligned}
&\sum_{k=1}^{n} b_k\frac{\omega_n(z)}{(z-z_k)\omega_n'(z_k)}\\
&\quad=\sum_{k=1}^{n} b_k\frac{w_k\Psi'(w_k)(x^n-1)}{n(\Psi(w)-\Psi(w_k))}\left(\frac{\Psi_n(w)}{\Psi_n(w_k)}-1\right)+\sum_{k=1}^{n} b_k\frac{w_k}{n}\,\frac{w^n-1}{w-w_k}\\
&\quad\quad+\sum_{k=1}^{n} b_k\frac{w_k}{n}(w^n-1)\left[\frac{\Psi'(w_k)}{\Psi(w)-\Psi(w_k)}-\frac{1}{w-w_k}\right]\\
&\quad:= I_5+I_6+I_7.
\end{aligned}$$

By applying (4.1) and (3.4) in Lemma 2, we obtain

$$\max_{z\in\overline{D}}|I_5| = O\left(\max_{1\le k\le n}|b_k|\right).$$

In order to estimate I_7, we may assume, without loss of generality, that $|w-1|$ is not greater than $|w-w_k|$, $k=1,2,\dots,n-1$, so that

$$|\arg w|\le\frac{\pi}{n}, \qquad \left|\frac{w^n-1}{w-1}\right|\le n,$$

and

$$\left|\frac{1}{w-w_k}\right| \le \begin{cases} \pi/n, & 1 \le k \le n/2, \\ (n-k)/k, & n/2 < k \le n-1. \end{cases}$$

Hence from (3.9), we have

$$\left| b_n \frac{w_n}{n}(w^n - 1)\left[\frac{\Psi(1)}{\Psi(w) - \Psi(1)} - \frac{1}{w-1}\right]\right| \le C|b_n| \tag{4.5}$$

and

$$\begin{aligned} &\left|\sum_{k=1}^{n-1} b_k \frac{w_k}{n}(w^n - 1)\left[\frac{\Psi'(w_k)}{\Psi(w) - \Psi(w_k)} - \frac{1}{w-w_k}\right]\right| \\ &\le C \max_{1\le k\le n-1} |b_k| \frac{1}{n}\sum_{k=1}^{n-1} \frac{\sigma_1(|w-w_k|)}{|w-w_k|} \\ &\le C \max_{1\le k\le n-1} |b_k| \frac{1}{n}\sum_{k=1}^{n} \frac{\sigma_1(k/n)}{k/n} \\ &\le C \max_{1\le k\le n-1} |b_k| \int_0^1 \frac{\sigma_1(t)}{t} dt \le C \max_{1\le k\le n-1} |b_k|, \end{aligned} \tag{4.6}$$

where the condition in (2.2) is used. Thus combining (4.5) and (4.6), we obtain

$$\max_{|z|\in\overline{D}} |I_7| = O\left(\max_{1\le k\le n} |b_k|\right).$$

Finally, by the Marcinkiewicz-Zygmund inequality for $1 < p < \infty$ (cf. [15]), we have

$$\|I_6\|_p = O\left(\frac{1}{n}\sum_{k=1}^{n} |b_k|^p\right)^{1/p} = O\left(\max_{1\le k\le n} |b_k|\right).$$

This completes the proof of the lemma. □

We now return to the estimates of (2.13) and (2.14) and obtain

$$\begin{aligned} A_{kj}(z) &= \left(\frac{\omega_n(z)}{(z-z_k)}\right)^{q+1} \frac{1}{j!}\alpha_{k\ q-j}(z-z_k)^q \\ &\quad + \left(\frac{\omega_n(z)}{z-z_k}\right)^{q+1} \frac{(z-z_k)^j}{j!} \sum_{\nu=0}^{q-j-1} \alpha_{k\nu}(z-z_k)^\nu \\ &:= I_8(z) + I_9(z). \end{aligned}$$

Hence, by (3.1) in Lemma 1, (3.4) in Lemma 2, (2.5), (2.9) in Theorem 1, and (4.1), (4.3) in Lemma 4, we have

$$\sum_{k=1}^{n} |I_8(z)| = O\left(\frac{1}{n^j}\right) \max_{z\in\overline{D}} \sum_{k=1}^{n} \left|\frac{\omega_n(z)}{(z-z_k)\omega_n'(z_k)}\right| = O\left(\frac{\ln n}{n^j}\right)$$

and for $1 < p < +\infty$

$$\left\|\sum_{k=1}^{n} b_k I_q(z)\right\|_j = O(n) \max_{1\le k\le n} \left|b_k \alpha_{k\ q-j}\right| = O\left(\frac{1}{n^j}\right) \max_{1\le k\le n} |b_k|.$$

Similarly, by (3.1) in Lemma 1, (3.4) in Lemma 2, (2.5), (2.9) in Theorem 1, and (4.2) in Lemma 4, we have

$$\sum_{k=1}^{n} |I_9(z)| = O\left(\frac{1}{n^j}\right).$$

By combining these estimates, we have established both (2.13) and (2.14). □

5. Proof of Theorems 3–6

Let Γ belong to class j_2 and $f \in A(\overline{D})$. It is known (cf. Theorems 1 and 6 in [5, Chapter 9]) that there exists $P_N \in \pi_N$ such that

$$\max_{z\in\overline{D}} |f(z) - P_N(z)| = O\left(\omega\left(f; \frac{1}{N}\right)\right) \tag{5.1}$$

and

$$\max_{z\in\overline{D}} \left|P_N^{(m)}(z)\right| = O\left(N^m \omega\left(f; \frac{1}{N}\right)\right), \qquad m = 1, 2, \ldots. \tag{5.2}$$

By using the first part of Theorem 2, we have

$$f(z) - H_N(f; z) = f(z) - P_N(z) + \sum_{k=1}^{N} (P_N(z_k) - f(z_k))\, A_{k0}(z). \tag{5.3}$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{q} P_N^{(j)}(z_k) A_{kj}(z)$$

Hence, by applying (2.13) and (2.14) of Theorem 2 and (5.1), (5.2) above, we have completed the proof of Theorem 3. Next, we write

$$f(z) - \widetilde{H}_N(f; z) = f(z) - H_N(f; z) + \sum_{k=1}^{N} \sum_{j=1}^{q} a_k^{(j)} A_{kj}(z).$$

Here, by using the hypothesis (2.17) and Theorem 3, we have

$$\max_{z\in\overline{D}} |f(z) - H_N(f; z)| \to 0,$$

and by using the hypothesis (2.16) and applying (2.13) in Theorem 2, we also have

$$\max_{z\in\overline{D}} \left| \sum_{k=1}^{N} \sum_{j=1}^{q} a_k^{(j)} A_{kj}(z) \right| \to 0.$$

This completes the proof of Theorem 4. The proofs of Theorems 5 and 6 are similar simply by applying (2.14) in Theorem 2, noting that Hölder's inequality can be applied for $0 < p \le 1$ and using the result for $p = 2$. □

6. Final remarks

In this section, we give examples of the domain D whose boundary curve Γ belongs to classes j_1 and j_2. Let Γ be of class C^1 and denote its angle of inclination as a function of arc length s by $\theta(s)$, $0 \le s \le |\Gamma|$, the length of Γ.

Proposition 1. *If* Γ *satisfies*

$$\int_0^a \frac{\omega(\theta\,;\,t)}{t}|\ln t|^3\,dt < \infty, \qquad a>0, \tag{6.1}$$

then Γ *belongs to class* j_1.

Of course, every Γ of class $C^{1+\delta}$ for some $\delta > 0$ satisfies (6.1). We also have the following

Proposition 2. *If* Γ *satisfies*

$$\int_0^a \frac{\omega(\theta'\,;\,t)}{t}\,|\ln t|^2\,dt < \infty, \qquad a>0, \tag{6.2}$$

then Γ *belongs to class* j_2.

Of course, every Γ of class $C^{2+\delta}$ for some $\delta > 0$ satisfies (6.2).

To prove these results, we need the following result in [14]: If

$$\int_0^a \frac{\omega(\theta^{(n)}\,;\,t)}{t}\,dt < \infty, \qquad a>0, \tag{6.3}$$

then $\Psi^{(n+1)}$ is continuous on $|w| \geq 1$ and

$$\omega\left(\Psi^{(n+1)}\,;\,t\right) = O\left(\int_0^t \frac{\omega(\theta^{(n)}\,;\,\tau)}{\tau}\,d\tau + t\int_t^a \frac{\omega(\theta^{(n)}\,;\,\tau)}{\tau^2}\,d\tau + t\ln\frac{1}{t}\right), \quad a>0. \tag{6.4}$$

Let $n = 0$. If (6.1) is satisfied, so is (6.3), and hence Ψ' is continuous on $|w| \geq 1$. Using (6.4) for $n = 0$, we have

$$\begin{aligned}
&\int_0^a \frac{\omega(\Psi'\,;\,t)}{t}\,|\ln t|^2\,dt \\
&= O\Bigg(\int_0^a\left(\int_0^t \frac{\omega(\theta\,;\,\tau)}{\tau}\,d\tau\right)\frac{|\ln t|^2}{t}\,dt \\
&\qquad + \int_0^a\left(t\int_t^a \frac{\omega(\theta\,;\,\tau)}{\tau^2}\,d\tau\right)\frac{|\ln t|^2}{t}\,dt + \int_0^a \frac{t\,\ln(1/t)}{t}|\ln\,t|^2\,dt\Bigg) \\
&= O\Bigg(\int_0^a\left(\int_\tau^a \frac{|\ln\,t|^2}{t}\,dt\right)\frac{\omega(\theta\,;\,\tau)}{\tau}\,d\tau \\
&\qquad + \int_0^a\left(\int_0^\tau |\ln t|^2\,dt\right)\frac{\omega(\theta\,;\,\tau)}{\tau^2}\,d\tau\Bigg) + O(1) \\
&= O\left(\int_0^a \frac{\omega(\theta\,;\,t)}{t}|\ln t|^3\,dt\right) + O(1) < \infty.
\end{aligned}$$

That is, Γ is of class j_1. This completes the proof of Proposition 1. The proof of Proposition 2 is similar by applying $n = 1$ in (6.3) and (6.4), using the condition (6.2). □

We conclude this paper by posing three open problems.

(1) In this paper, we consider the $(0, 1, \dots, q)$ Hermite-Fejér interpolation problem where the interpolatory polynomials $H_N(f; \cdot)$ satisfy $H_N^{(j)}(f; z_k) = 0$, $j = 1, \dots, q$ and $k = 1, \dots, n$. It is interesting to study if the convergence and estimates in this paper are still valid if we impose a more general interpolatory condition:

$$H_N^{(j)}(f; z_k) = 0 \quad \text{for } j = 1, \dots, q_k, \; k = 1, \dots, n,$$

where $q_k = q_k(n)$ satisfies $\max_{1 \le k \le n} q_k(n) \le M < \infty$ for all n.

(2) How much can the Fejér points $z_k = z_{nk}$ be perturbed on Γ so that the convergence and estimates in this paper are still valid?

(3) If D is different from the unit disk, do there exist $(0, m_1, \dots, m_q)$ Birkhoff-Fejér interpolants $B_N(f; \cdot)$; that is,

$$B_N(f; z_k) = f(z_k) \quad \text{and} \quad B_N^{(m_j)}(f; z_k) = 0,$$

for $j = 1, \dots, q$ and $k = 1, \dots, n$? If $B_N(f; \cdot)$ exist, do they converge to f in L^p, $0 < p < \infty$? For the unit disk, results on convergence and estimates have been obtained in [10].

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